OPTIMIZING THE MAGNITUDE RESPONSE OF MATCHED Z-TRANSFORM FILTERS (“MZTI”) FOR LOUDSPEAKER EQUALIZATION

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The magnitude response of matched z-transform (MZT) filters deviates from the response of analog filters in a relatively gradual fashion. The error can be minimized by adding zeros to an all-pole filter or by shifting the position of the poles or zeros in a biquadratic filter. A set of efficient closed form algorithms is presented which provide optimum coefficients. The resulting filters (dubbed “MZTi”) display excellent agreement with the analog target filters, both in their magnitude and phase responses.

INTRODUCTION

Most digital filters employed in the equalization of loudspeakers have been derived using the bilinear z-transform (BZT). The BZT method seeks to provide a digital approximation of an analog filter which is guaranteed to be stable and which exhibits no image distortion due to reflection about the Nyquist frequency. While it achieves these goals, it does so by introducing response distortions that are often larger than the image distortions it seeks to avoid. It could be said that the cure it provides is worse than the disease it treats.

The professional audio industry’s near universal reliance on the BZT method can be attributed to both convenience and ignorance. The BZT method offers a closed-form, computationally inexpensive solution, which produces useable, though often imprecise, approximations of all common filter types. In short, it is a convenient solution. Its lack of precision has been tolerated because few end users are aware of its shortcomings, and even fewer are aware that better performance is achievable.

More precise filters have been created using variations of the BZT method [3], and using various methods to improve the accuracy of matched z-transform (MZT) derived filters [4]. Unfortunately, no MZT improvement algorithm has been publicly disclosed which offers a closed-form solution with broad applicability. Iterative methods can provide highly optimized results, but are too computationally expensive to be used for real-time coefficient calculations.

This paper presents a set of closed-form solutions that can be applied to each of the filter types that are commonly implemented with biquadratic IIR (Infinite Impulse Response) filters. The approach in each case is to first calculate some or all of the filter coefficients using the standard MZT method, then to calculate the difference in response between the ideal analog filter and the MZT filter. Finally, a closed-form calculation is used to determine the coefficients of a filter which exactly matches the target magnitude response at selected frequencies.

Filters derived using the particular set of techniques presented here have been dubbed “MZTi” filters, for “matched z-transform, improved”.

1 BACKGROUND

1.1 Optimization Criteria

The methodologies presented in this paper can be generalized to any application. The primary aspect that renders them specialized to loudspeaker equalization is the particular choice of optimization criteria.

Digital filters applied to loudspeakers are nearly always equalization filters. After application of these filters, the magnitude response of the system will typically be nearly flat. Consequently, it is not a good assumption that the accuracy of the response of a bell filter is more critical at its peak than in the rising response region of its response. On the contrary, if a high frequency boost filter matches well at its peak, but not in the rising response leading up to the peak, the error in the net response will be much greater, and much more audible, than if the rising response were optimized at the expense of matching the peak boost.

The Orfanidis BZT method [3], while a significant improvement over the standard BZT approach, optimizes the width of the bell at its –3 dB points and the response at the Nyquist frequency. Unfortunately, the response on the low frequency side of the bell often deviates significantly from ideal. And, of course, the necessary antialias filter drives the response of the system toward zero at Nyquist, rendering moot the precise match of the filter at that frequency.

The general rule that was applied during the development of the MZTi methods was this: never sacrifice lower frequency precision in order to obtain higher frequency precision. Any response deviation at 6 kHz will be much more audibly significant than a similar error at 20 kHz. The methods presented herein employ an exact match at very low frequencies and seek to match the ideal response as closely as possible and as high in frequency as possible, finally, allowing the filter to diverge slightly from ideal as the Nyquist frequency is approached.

1.2 Bell Filters

One of the most common applications of digital filters in audio is the bell filter, which is often referred to, somewhat imprecisely, as a "parametric filter". The first sweepable-frequency, or parametric, filters happened to be bell filters. Hence the label, "parametric filter", became closely associated with bell filters; even though shelf filters, bandpass filters, and even high-pass and low-pass filters are parametric, as long as they are adjustable. A bell filter is most commonly realized with two poles and two zeros - in other words, with a fully utilized, biquadratic filter section.

There is no universal definition of the parameters of a bell filter. The analog response always takes the form:

$$H(s) = \frac{s^2 + \frac{g_0 \cdot s}{Q} + 1}{s^2 + \frac{s}{Q} + 1}, \tag{1}$$

but a constant Q produces widely varying bell width, and does not give complementary filters (i.e., a boost filter is not the inverse of a cut filter with the same Q). Many practitioners in the professional audio field have adopted an adjustment to Q which provides complementary filters with a more-or-less constant bell width. If g_0 is the gain at the center frequency, and given a nominal Q, called Q_{nom}, calculate an adjusted Q according to:

If $g_0 > 2$,

$$Q = Q_{nom} * g_0 \Big/ \sqrt{g_0^{\,2} - 2}. \tag{2}$$

If $0.5 < g_0 < 2$,

$$Q = Q_{nom} * \sqrt{g_0}. \tag{3}$$

If $g_0 < 2$,

$$Q = Q_{nom} * g_0 \Big/ \sqrt{1 - 2g_0^{\,2}}. \tag{4}$$

The BZT implementation of a bell filter works well with center frequencies below 1/2-Nyquist, but the response deviates significantly from ideal with higher center frequencies. As the center frequency approaches Nyquist, the BZT method "pre-warps" the center frequency and Q of the filter so that the center frequency and width of the bell is approximately preserved. However, the resulting filter response deviates from ideal, slightly in the rising response region and significantly at high frequencies. Regardless of the target filter shape, the response of a BZT bell filter returns to unity at Nyquist. Consequently, a BZT-derived filter will poorly emulate any bell filter with a response that should not return to unity at Nyquist. Figure 1 illustrates an example of such a filter.

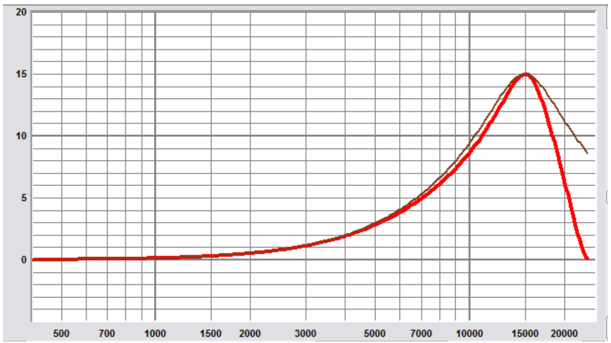

Figure 1: Analog vs. BZT Bell Filter, 15 kHz, +15 dB, Q = 2.0, SR = 48 kHz

The Orfanidis method reduces the deviation from ideal near Nyquist, but exacerbates the problem in the rising response region.

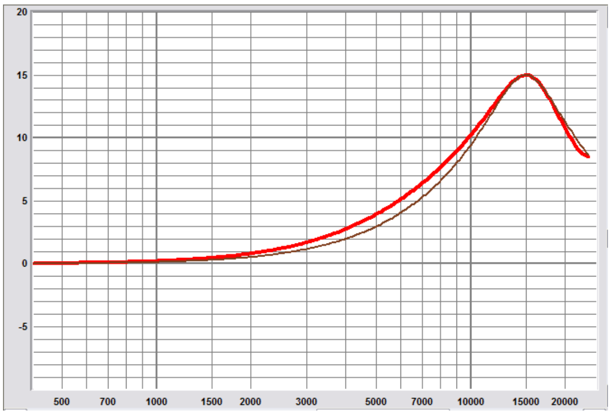

Figure 2: Analog vs. Orfanidis BZT Bell Filter, 15 kHz, +15 dB, Q = 2.0, SR = 48 kHz

Orfanidis suggested several alternative methods for optimizing the shape of bell filters. If we specify the lower band edge frequency and calculate the upper band edge frequency using the pre-warped geometric mean rule [3, sec. 5], the response matches the target response much better in the rising response region. This particular variation is the BZT-based filter that best meets the optimization criteria we have established for loudspeaker equalization.

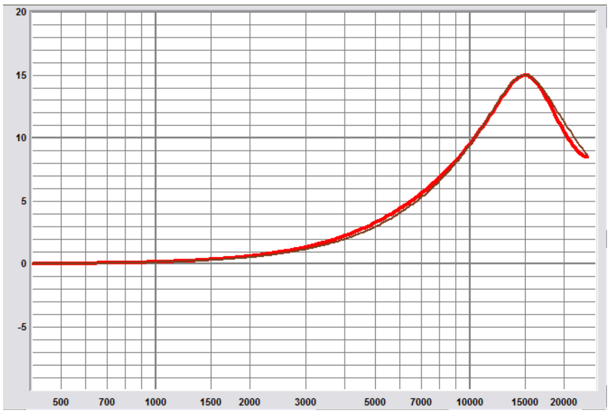

Figure 3: Analog vs. Orfanidis BZT Bell Filter, with pre-warped f2

The standard MZT approach produces a filter which gradually departs from ideal as frequency increases. For low-Q, high frequency filters, the discrepancy may be significant. The example in figure 4 shows a 1.5 dB overshoot at the peak of the filter, and a 4.5 dB difference at Nyquist.

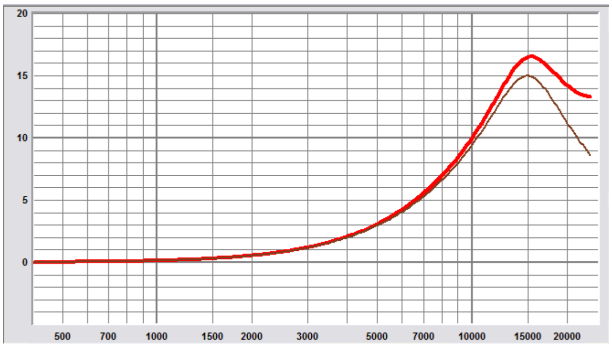

Figure 4: Analog vs. Standard MZT Bell Filter, 15 kHz, +15 dB, Q = 2.0

A more precise bell filter is of particular interest and value and the MZT approach is an attractive starting point for two reasons. It preserves the resonant frequency and decay of the target filter, and it has a particularly benign error function.

1.3 Error Functions

To identify potential means of improving the precision of digital filters, it is instructive to assess the characteristics of their error functions - the difference between the response of an analog target filter and the response of a candidate digital filter. The figures below compare the response and error functions of BZT and MZT realizations of two particular filters. The filters are a high frequency, underdamped, 2nd order low-pass filter and a high frequency bell filter with 15 dB of center frequency gain and moderate Q. They were chosen specifically because they represent a worst case: standard filter methods do not emulate them very well.

The BZT method doesn't work well at all for an underdamped low-pass filter. The frequency and height of the response peak is correct, but its width exhibits severe distortion caused by frequency warping. Parametric pre-warping is of no help, because changing the Q of the filter also changes the height of the peak. The BZT bell filter matches a little better, but is still noticeably deficient.

Note the shape of the error functions (the bold curves). In both BZT examples the error function has two inflection points. A correction filter would have to be quite complex to correct the response with any degree of precision.

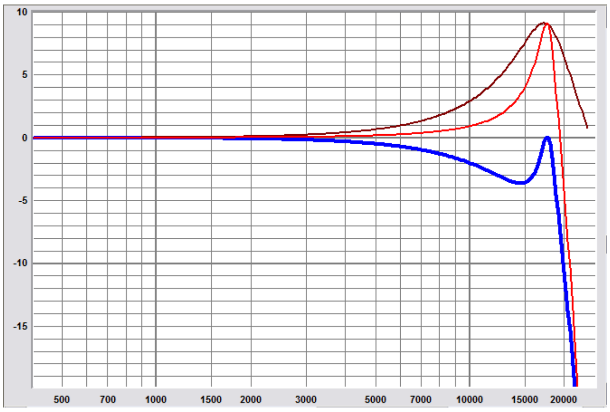

Figure 5: Error Function, BZT Low-pass Filter, 18 kHz, Q=2.8

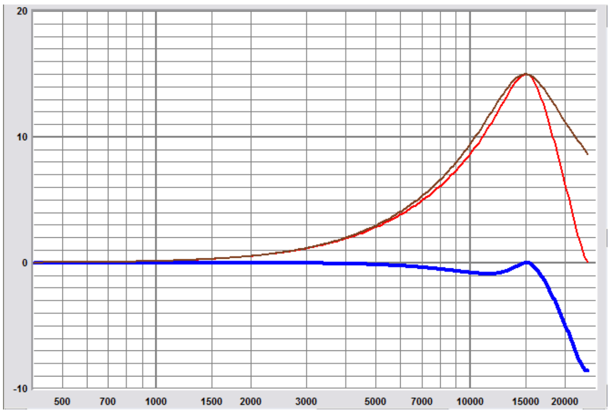

Figure 6: Error Function, BZT Bell Filter, 15 kHz, +15 dB, Q = 2.0

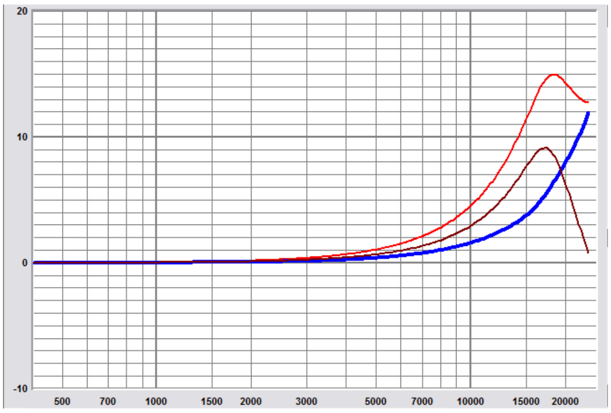

Figure 7: Error Function, MZT Low-pass Filter, 18 kHz, Q=2.8

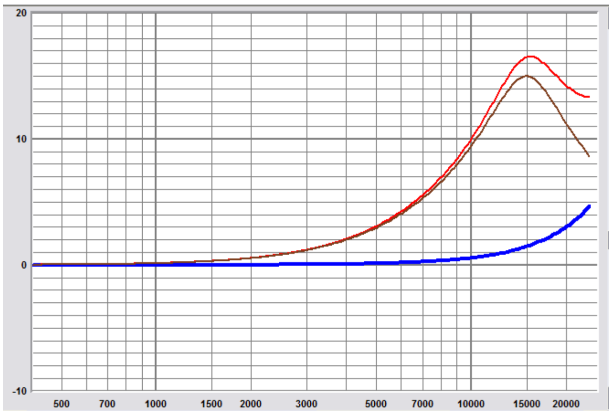

Figure 8: Error Function, Analog vs. MZT Response, 15 kHz Bell Filter (Q = 2.0)

The MZT filters exhibit similar or greater amounts of error, but the error functions are much simpler. We should be able to improve the response significantly with relatively simple filters. All that is required is a closed-form solution to determine a "best" correction filter.

An algorithm has been developed which provides a digital filter whose analog magnitude response exactly matches a target magnitude response at selected frequencies. It is a closed-form approach and is relatively inexpensive, computationally. This algorithm will be used in all of the MZTi calculations presented in this paper.

2 AN FIR CURVE FIT TRANSFORM

The analog frequency response of a digital filter can be calculated directly by substituting $e^{-j\omega T}$ for z^{-1} in the z-domain expression for the filter. Given a specification of a target filter at a small number of frequencies, a system of equations can be generated. The solution of this system of equations gives the coefficients of a digital filter that matches the response of the target filter at the specified frequencies.

A single-zero filter can be completely specified with two values (b_0 and b_1, or g and b_1), so we only need the response of the target filter at two frequencies - for example, DC and Nyquist. However, as we discussed earlier, Nyquist is not an optimum defining point for loudspeaker equalization filters. We will show that using a lower frequency as the second defining point provides results in accordance with the stated optimization criteria.

Similarly, a two-zero filter can be completely specified with three values (b_0, b_1, and b_2). To solve a system of equations for a 2nd order correction filter, we will need three frequency response points.

The closed form solution of the system of equations is greatly simplified if the frequencies selected are rational fractions of the Nyquist frequency. Using DC and 1/2-Nyquist for the one-zero case, and DC, 1/3- and 2/3-Nyquist for the two-zero case, results in surprisingly simple expressions for the response of the digital filter. It also provides correction filters with the desired nature - they match exactly at low to mid frequencies and diverge from ideal only as Nyquist is approached. Because the expressions are simple, the calculation of the error functions is inexpensive.

To derive an error function, the response of a "candidate" digital filter must be calculated. In the case of a biquadratic filter, substituting $e^{-j\omega T}$ for z^{-1} in the z-domain expression for the filter results in:

$$H(\omega) = g\,\frac{1 + b_1 e^{-j\omega T} + b_2 e^{-2j\omega T}}{1 + a_1 e^{-j\omega T} + a_2 e^{-2j\omega T}}.$$

At 1/2-Nyquist, $e^{-j\omega T} = -j$, so the analog response of a digital biquadratic filter is simply:

$$H_D(N/2) = g\,\frac{1 - jb_1 - b_2}{1 - ja_1 - a_2}. \qquad (5)$$

At 1/3-Nyquist, $e^{-j\omega T} = .5 - 0.866j$, so the response of the filter is:

$$H_D(N/3) = g\,\frac{1 + b_1(.5 - .866j) - b_2(.5 + .866j)}{1 + a_1(.5 - .866j) - a_2(.5 + .866j)}. \qquad (6)$$

At 2/3-Nyquist, $e^{-j\omega T} = -.5 - 0.866j$, so the response of the filter is:

$$H_D(2N/3) = g\,\frac{1 - b_1(.5 + .866j) - b_2(.5 - .866j)}{1 - a_1(.5 + .866j) - a_2(.5 - .866j)}. \qquad (7)$$

The magnitude responses are considerably simpler:

$$|H_D(N/2)| = g\sqrt{\frac{1 - 2b_2 + b_2^2 + b_1^2}{1 - 2a_2 + a_2^2 + a_1^2}}, \qquad (8)$$

$$|H_D(f_N/3)| = g\frac{\sqrt{1+b_1-b_2+b_1^2+b_1b_2+b_2^2}}{\sqrt{1+a_1-a_2+a_1^2+a_1a_2+a_2^2}}, \quad (9)$$

$$|H_D(2f_N/3)| = g\frac{\sqrt{1-b_1-b_2+b_1^2-b_1b_2+b_2^2}}{\sqrt{1-a_1-a_2+a_1^2-a_1a_2+a_2^2}}. \quad (10)$$

A target correction filter may be obtained by dividing the desired response at the cardinal frequencies by these relatively simple expressions.

Given a target frequency response specification at DC (H_0) and 1/2-Nyquist (H_1), the coefficients of a 1st-order FIR (Finite Impulse Response) filter can be calculated as:

$$b_0 = \frac{H_0(1+\sqrt{2(H_1)^2-1})}{2},$$

$$b_1 = H_0 - b_0.$$

Given the target frequency response specification at DC (H_0), 1/3-Nyquist (H_1), and 2/3-Nyquist (H_2) the coefficients of a 2nd-order FIR filter can be calculated as:

$$b_1 = \frac{H_0 - \sqrt{H_0^2 - 2H_1^2 + 2H_2^2}}{2},$$

$$b_2 = \frac{3(H_0 - b_1) - \sqrt{-3H_0^2 + 12H_1^2 - 6H_0b_1 - 3b_1^2}}{6},$$

$$b_0 = H_0 - b_1 - b_2.$$

If the resulting coefficients are real, then they specify a filter which matches the target response exactly at the specified frequencies. If the target response is not achievable, b_1 and/or b_2 will be complex. In that case, an approximate curve fit may be obtained by using the magnitude of the complex value as the coefficient. For the filter derivations defined in this paper, the coefficients have always been real.

For a detailed derivation of the FIR curve fit transform, please refer to the appendix.

3 MZTi FILTERS

The above technique can be used to improve MZT filters. In this paper we will present a solution for a variable-Q low-pass filter, which is useful in and of itself, and can also be applied to any filter with an IIR component. We will also present solutions for bell filters employing varying numbers of coefficients.

3.1 MZTi 2nd-Order Low-pass Filter

A low-pass filter is an all-pole filter. Consequently, in a typical biquadratic MZT implementation, only the poles, or "a" coefficients, are utilized. The zeros, or "b" coefficients, are available for de-aliasing. We'll show how to develop both a one-zero and a two-zero correction filter using the approach given above.

The normalized ($\omega_0 = 1$) response of a variable-Q, 2nd-order low-pass filter is:

$$H_A(s) = \frac{1}{s^2 + \frac{s}{Q} + 1}. \quad (11)$$

Applying the standard MZT methodology [1, 2], the "a" coefficients can be calculated as follows:

If Q > 0.5, the IIR coefficients are:

$$a_1 = -2e^{(\frac{-\omega_0 T}{2Q})}\cos(\sqrt{1-(\frac{1}{2Q})^2}\,\omega_0 T), \quad (12)$$

$$a_2 = e^{(\frac{-\omega_0 T}{Q})}. \quad (13)$$

If Q < 0.5, the IIR coefficients are:

$$a_1 = -(e^{-\omega_0 T(\frac{1}{2Q}+\sqrt{(\frac{1}{2Q})^2-1})} + e^{-\omega_0 T(\frac{1}{2Q}-\sqrt{(\frac{1}{2Q})^2-1})}), \quad (14)$$

$$a_2 = e^{(\frac{-\omega_0 T}{Q})}. \quad (15)$$

The FIR coefficients (b_0, b_1 and b_2), are available for de-aliasing. To determine the optimum coefficients, we evaluate the difference between the gain of the analog filter and the all-pole digital filter.

For a single-zero de-aliasing filter, compare the response of the digital, MZT filter to the analog target filter at DC and 1/2-Nyquist ($f_N/2$ or $\omega_N/2$). To find the 1/2-Nyquist gain of the analog target filter, substitute $jf_N/2f_0$ for s in the system equation above. Then calculate the analog gain magnitude,

$$g_A = \left|H(j\, {f_N}/{2f_0})\right| = \frac{1}{\sqrt{\left(1-\left(\frac{f_N}{2f_0}\right)^2\right)^2 + \left(\frac{f_N}{2f_0}\right)^2}}. \quad (16)$$

The gain of the digital filter at 1/2-Nyquist is

$$g_D = \left|\frac{1}{1-a_2-ja_1}\right|. \quad (17)$$

The coefficients of a single-zero filter to correct for the difference are:

$$b_0 = \frac{1+\sqrt{2(\frac{g_A}{g_D})^2 - 1}}{2}, \quad (18)$$

$$b_1 = 1 - b_0. \quad (19)$$

To obtain a two-zero, de-aliasing filter, calculate the response of the target correction filter at 1/3- and 2/3-Nyquist. The magnitude response of the digital filter at 1/3-Nyquist is:

$$|H_d(f_N/3)| = \frac{1}{\sqrt{1+a_1-a_2+a_1^2+a_1a_2+a_2^2}}. \quad (21)$$

At 2/3-Nyquist, the response is:

$$|H_d(2f_N/3)| = \frac{1}{\sqrt{1-a_1-a_2+a_1^2-a_1a_2+a_2^2}}. \quad (23)$$

The target correction filter is defined at three frequencies:

$$H_0 = 1/H_d(0), \quad (24)$$

$$H_1 = \left| H_a(\frac{f_N}{3f_0}) \middle/ H_d(\frac{f_N}{3}) \right|, \quad (25)$$

$$H_2 = \left| H_a(\frac{2f_N}{3f_0}) \middle/ H_d(\frac{2f_N}{3}) \right|. \quad (26)$$

The two-zero filter coefficients to provide this filter are:

$$b_1 = \frac{H_0 - \sqrt{H_0^2 - 2H_1^2 + 2H_2^2}}{2}, \quad (27)$$

$$b_2 = \frac{3(H_0 - b_1) - \sqrt{-3H_0^2 + 12H_1^2 - 6H_0b_1 - 3b_1^2}}{6}, \quad (28)$$

$$b_0 = H_0 - b_1 - b_2. \quad (29)$$

Figure 9 shows the response of an MZTi low-pass filter, compared to the analog target filter. The bold curve is the error function. Note that the only discrepancy occurs at very high frequencies. The response matches very well up to 20 kHz.

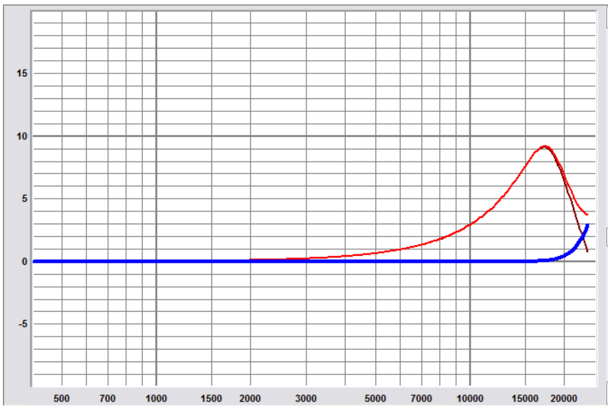

Figure 9: MZTi Low-pass Filter, Single-Zero Correction Filter, 18 kHz, Q = 2.8

Higher-order, low-pass filters can be synthesized by applying the 2nd-order, low-pass methodology to each pair of poles.

3.2 MZTi Bell Filters

The normalized response of a bell filter,

$$H(s) = \frac{s^2 + \frac{g_0 \cdot s}{Q} + 1}{s^2 + \frac{s}{Q} + 1}, \quad (30)$$

can also be represented as:

$$H(s) = \frac{A(s)}{B(s)} = \frac{1}{B(s)} \frac{1}{1/A(s)}, \quad (31)$$

where

$$A(s) = s^2 + \frac{g_0 \cdot s}{Q} + 1, \quad (32)$$

and

$$B(s) = s^2 + \frac{s}{Q} + 1. \quad (33)$$

The partial filters, 1/B(s) and 1/A(s), are each 2nd-order, low-pass filters. The resonant frequency of each is the center frequency of the parametric filter. The Q of 1/B(s) is Q. The Q of 1/A(s) is Q/g_0.

3.2.1 Bicubic and Biquartic MZTi Bell Filters

Using the corrected 2nd-order, low-pass filter algorithm in section 3.1, the coefficients for the two partial filters can be calculated with de-aliasing. Then, the partial filters can be combined to form a complete bell filter.

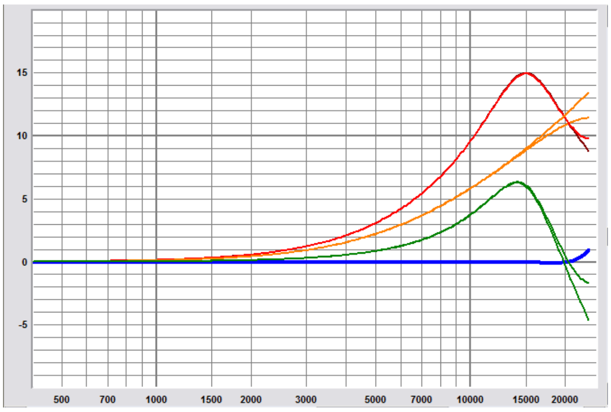

Figure 10: Bicubic MZTi Bell Filter And Partial Filters, 15 kHz, +15 dB, Q = 2.0

An example illustrates how this approach minimizes the error near Nyquist. The 15 kHz, 15 dB boost (gain = 5.623) filter with a Q of 2.0 can be synthesized by calculating the coefficients for two 2^{nd}-order, low-pass filters, one with a Q of 2.0 and an inverted one with a Q of 2/5.623 or 0.355. The responses of the two MZTi partial filters, as well as the net parametric boost filter, are shown in Figure 10 compared to the corresponding ideal filters. Note that the deviation of the two partial filters from ideal is very similar, but in opposite directions. When the filters are combined, the deviations nearly cancel; giving a net result that is very nearly ideal - with only a 1 dB error at Nyquist.

The 2^{nd}-order, low-pass calculations used to determine the coefficients for the two low-pass filters employed a single-zero correction. Consequently, the method always returns the value of b_2 as zero. After $1/A(s)$ is inverted, its a_2 coefficient is zero. The complete filter can be implemented with three "a" coefficients and three "b" coefficients. This can be termed a "bicubic" implementation:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}. \qquad (34)$$

The same procedure can be carried out using two-zero correction of the low-pass filters. In this case, each partial filter consumes a complete biquadratic section. The net result is a biquartic or dual-biquad solution. When a digital signal processing platform is based on biquadratic filters, there may be cases where two biquads would be used even if the bicubic solution is satisfactory. In those cases, there is no reason not to implement a biquartic filter.

3.2.2 Biquadratic MZTi Bell Filter

A bell filter implemented as a single biquadratic filter requires all of its a and b coefficients to model the analog poles and zeros, so there are no coefficients left over for de-aliasing. Consequently, the only correction that can be realized without increasing the order of the filter is to shift the zeroes and/or poles.

Of the several methods that have been proposed for determining a helpful shift of the zeros of an MZT filter [4], all have limited effectiveness. Shifting a real zero can reduce overshoot, but exhibits a poorly matched response in the rising response region. Methods of shifting the complex zeros have only been effective over a limited range of conditions.

From Clark/Ifeachor [4], "No generic shifting principle was found for complex roots. Complex-root shifting methods tended to produce response distortion at particular root positions, produce non-conjugate roots, or not produce a closed-form solution for the shifting factor." Parametric pre-warping tended to improve the accuracy of the response at a specific frequency (i.e., center frequency gain) while sacrificing the accuracy of the response at other frequencies.

A new algorithm is presented here which precisely matches the analog magnitude response up to 2/3-Nyquist, then deviates only slightly at frequencies approaching Nyquist.

As discussed earlier, a bell filter can be represented as the quotient of a pair of 2^{nd}-order, low-pass filters. If the higher Q filter is in the numerator, the result is a boost filter. Conversely, if the lower Q filter is in the numerator, the result is a cut filter. Because a high Q filter has an inherently longer resonant decay than a low Q filter, the impulse response of a bell filter is most strongly affected by the higher Q part of the filter. Therefore, to achieve the most precise impulse response, we will retain the MZT-derived coefficients for the higher Q half of the filter and shift the lower Q half of the filter. Consequently, we will be shifting the zeros of boost filters, but we will be shifting the poles of cut filters. The algorithm produces minimum phase filters, so it may be simplified by only implementing the boost filter case. When a cut filter is encountered, derive the digital filter for the complementary boost filter, then invert the result.

The coefficients to be "shifted" will actually be calculated directly, using the FIR curve fit transform given above. In this case, however, the ideal "correction filter" includes both the inverted error function of the high-Q half of the filter and the analog response of the low-Q half of the filter. This is obtained simply by dividing the analog filter response by the response of the MZT-derived, high-Q half of the bell filter (at 1/3- and 2/3-Nyquist). Presented as a step-wise process, the single-biquad, MZTi bell filter algorithm is as follows:

1) If center frequency gain (g_0) is less than 1, set $g_0 = 1/g_0$. In other words, convert a cut filter specification to a boost filter specification.

2) Calculate the "a" coefficients for the filter, using the standard MZT method.

3) Calculate the magnitude response of the analog target filter and the magnitude response of the all-pole digital filter at DC, 1/3-, and 2/3-Nyquist. Divide the analog response magnitude by the digital response magnitude to obtain a three-frequency "correction filter" specification.

4) Use the FIR curve fit transform to determine the optimum, minimum phase, two-zero, correction filter coefficients: b_0, b_1, and b_2.

5) If the original filter was a cut filter, invert the digital filter.

Figure 11 shows the response of the worst case bell filter, calculated as just described. Note that the agreement is nearly perfect up to 2/3-Nyquist. The error at 20 kHz is less than 1 dB.

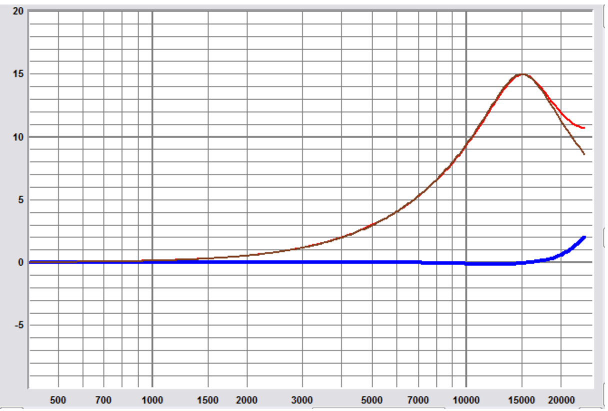

Figure 11: Biquadratic MZTi Bell Filter, 15 kHz, +15 dB, Q = 2.0

4 PHASE RESPONSE

MZT filters are inherently minimum phase. Filters designed using the FIR curve fit transform are also minimum phase by design. It would be expected that a minimum phase filter that more closely approximates its analog target's magnitude response will also more closely approximate its phase response.

This presumption is born out by comparisons of the phase response of various filters. As an example, figures 12 and 13 compare the phase response of a bell filter with the phase response of an analog target filter. With the exception of a fractional sample of delay, the MZTi phase response matches closely up to 20 kHz.

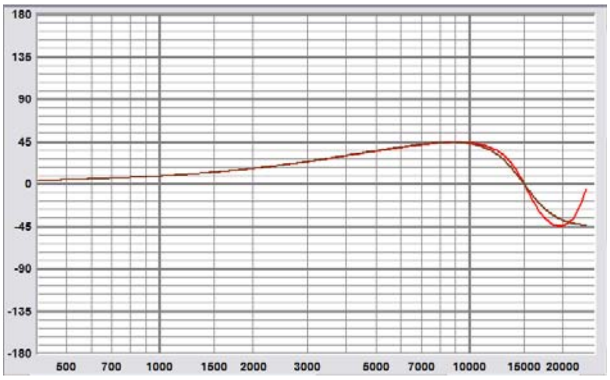

Figure 12: BZT Bell Filter Phase Response

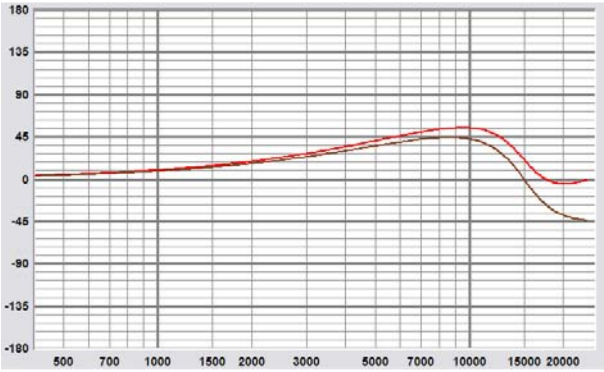

Figure 13: Biquadratic MZTi Bell Filter Phase Response

5 CONCLUSIONS

A set of closed-form filter development techniques was presented that improves the respone of MZT filters. Specific algorithms were presented for low-pass filters and bell filters. The same techniques may be applied, by extension, to other filter types.

The resultant filters exhibit magnitude and phase responses that closely match their analog target filters.

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APPENDIX - DERIVATION OF THE FIR CURVE FIT TRANSFORM

The frequency response of a normalized 2nd order FIR system, $h(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$, is given by:

$$H(\omega) = b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega}$$
$$= (b_0 + b_1 \cos\omega + b_2 \cos 2\omega) - j(b_1 \sin\omega + b_2 \sin 2\omega).$$

From the above relation we get

$$|H(\omega)|^2 = b_0^2 + b_1^2 + b_2^2 + 2b_0 b_1 \cos\omega + 2b_0 b_2 \cos 2\omega + 2b_1 b_2 \cos\omega \quad (35)$$

We compute the correction filter by matching the frequency response at three different frequency points $f_0 = 0$, $f_1 = \omega$, and $f_2 = \pi - \omega$, where ω is any arbitrary frequency point less than Nyquist. These frequencies are selected as they result in relatively simple equations when evaluated on (35). Let $k_0 = |H(0)|$, $k_1 = |H(\omega)|$, and $k_2 = |H(\pi - \omega)|$. So,

$$k_0 = b_0 + b_1 + b_2 \quad (36)$$

$$k_1^2 = b_0^2 + b_1^2 + b_2^2 + 2b_0 b_1 \cos\omega + 2b_0 b_2 \cos 2\omega + 2b_1 b_2 \cos\omega \quad (37)$$

$$k_2^2 = b_0^2 + b_1^2 + b_2^2 - 2b_0 b_1 \cos\omega + 2b_0 b_2 \cos 2\omega - 2b_1 b_2 \cos\omega \quad (38)$$

We now solve for b_1 in the above system of equations. Subtracting (38) from (37) results in:

$$k_1^2 - k_2^2 = 4b_0 b_1 \cos\omega + 4b_1 b_2 \cos\omega = 4b_1(b_0 + b_2)\cos\omega. \quad (39)$$

From (36), $b_0 + b_2 = k_0 - b_1$. Substituting this in (39) we get

$$a = b_1(k_0 - b_1) = b_1 k_0 - b_1^2,$$

where $a = (k_1^2 - k_2^2)/4\cos\omega$. Therefore,

$$b_1^2 - k_0 b_1 + a = 0 \cdot$$

Solving the above quadratic equation we get two values for b_1.

$$b_1 = \frac{k_0 \pm \sqrt{k_0^2 - 4a}}{2} = \{x_1, x_2\} \quad (40)$$

We now solve for b_2 with $b_1 = x_1$. From (35) we get

$$y_1 = b_0 + b_2, \quad (41)$$

where $y_1 = k_0 - x_1$. Substituting $b_1 = x_1$ and $b_0 = y_1 - b_2$ in (37) we get

$$k_1^2 = (y_1 - b_2)^2 + x_1^2 + b_2^2 + 2(y_1 - b_2)x_1 \cos\omega + 2(y_1 - b_2)b_2 \cos 2\omega$$
$$+ 2x_1 b_2 \cos\omega$$
$$= b_2^2(2 - 2\cos 2\omega) + b_2(2y_1 \cos 2\omega - 2y_1) + (y_1^2 + x_1^2 + 2y_1 x_1 \cos\omega)$$

We compute b_2 by solving the above quadratic equation.

$$b_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$

where

$$\alpha = 2(1 - \cos 2\omega)$$
$$\beta = 2y_1(\cos 2\omega - 1)$$
$$\gamma = y_1^2 + x_1^2 + 2y_1 x_1 \cos\omega - k_1^2$$

Each value of b_1 results in two different values for b_2. After we compute b_1 and b_2 we can compute b_0 from (36).

This procedure results in four different solutions for $\{b_0, b_1, b_2\}$. Some or all of the solutions may be complex. Of the real solutions, all have the same magnitude response, but the phase responses are different. Because the filter may need to be inverted later in the algorithm, only the minimum phase solution is acceptable. In the applications presented here, it was found, empirically, that the minimum phase filter was always returned by the following solutions:

$$b_1 = \frac{k_0 - \sqrt{k_0^2 - 4a}}{2},$$

$$b_2 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$

$$b_0 = k_0 - b_1 - b_2.$$